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On a modified version of a solvable ODE due to Painlevé

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Abstract
We introduce a modified version of an ODE originally introduced by Painlevé, and we show that it possesses a lot of periodic solutions. We also use (a simplified version of) it to verify the efficacy of an iterative approach used elsewhere, in more general contexts, to evince analogous properties of periodicity of (systems of) nonlinear ODEs.

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1. Introduction

At the beginning of last century Paul Painlevé (see [1], as quoted in [2]) introduced the nonlinear autonomous ODE

$$w'' = w'^2\,[(1-w^2)\,(1-k^2w^2)]^{-1/2} \times\{-\lambda^{-1} + [(1-w^2)\,(1-k^2w^2)]^{-1/2}\,w\,[2\,k^2w^2 - (1+k^2)]\} \tag{1.1}$$

and pointed out that it features the *general* solution

$$w(z) = \mathrm{sn}[\lambda\ \log(A\,z - B),\ k]. \tag{1.2}$$

1.1. Notation

$w \equiv w(z)$ is the dependent variable, z is the independent variable, primes indicate (here and throughout) differentiations with respect to the (complex) independent variable z, $w'(z) \equiv \mathrm{d}w\,(z)\,/\mathrm{d}z$, k and λ are arbitrary constants, $\mathrm{sn}(u,\ k)$ is the first Jacobian elliptic function (see, for instance, [3]), and $A,\ B$ are two arbitrary constants, the presence of which in (1.2) shows that this formula, (1.2), indeed provides the *general* solution of (1.1).

This ODE, (1.1), is a classical, much quoted, example of nonlinear ODE a simple local analysis of which only evidences poles as singularities in the finite part of the complex z-plane, but which in fact features, at the point $z = B/A$, an essential singularity at which the value of the solution itself is undefined; hence it is a (standard) example of an (autonomous!) nonlinear ODE that, in contrast to the expectation suggested by such a local analysis, does *not* possess the 'Painlevé property' to feature *only* meromorphic solutions, namely solutions the singularities of which in the (finite part of the) complex plane of the independent variable z are *only poles*.

The main purpose of the present paper is to exhibit the following modified version of (1):

$$\ddot{f} - \mathrm{i}\omega \dot{f} = \dot{f}^2 [(1-f^2)(1-k^2 f^2)]^{-1/2} \times \{-\lambda^{-1} + [(1-f^2)(1-k^2f^2)]^{-1/2} f [2k^2f^2 - (1+k^2)]\} \tag{1.3}$$

which features the additional *real* (indeed, without loss of generality, *positive*) constant ω, and of course reduces to (1.1), up to notational changes (see below), for $\omega = 0$ (but we hereafter assume $\omega > 0$). We hereafter consider the solutions $f \equiv f(t)$ of this modified ODE, (1.3), as functions of the new *real* independent variable t, and of course superimposed dots in (1.3) and throughout denote differentiations with respect to this variable t, i.e. $\dot{f}(t) \equiv \mathrm{d}f(t)/\mathrm{d}t$.

Our main result is to observe that this modified equation (1.3), features a lot of periodic solutions with period T,

$$T = 2\pi/\omega \tag{1.4}$$

$$f(t+T) = f(t) \tag{1.5}$$

for arbitrary values of k and λ, and moreover that *all* its *nonsingular* solutions are *periodic*, either with period T, see (1.4), or with a period $\tilde{T} = q\,T$ which is an integral multiple of T if the constants λ and k are related by either one of the following formulae:

$$2\pi\mathrm{i}\lambda = 4K(k)p/q \tag{1.6a}$$

$$2\pi\lambda = 2\,K'(k)\,p/q. \tag{1.6b}$$

Here and below $K(k)$ and $K'(k)$ are the complete elliptic integrals (in standard notation, see for instance [3]), and p, q are two coprime integers (without loss of generality we assume q to be *positive*).

In the following, merely for the ease of presentation, we refer to the new independent variable t as 'time', and we therefore consider (1.3) as an *evolution* ODE in *real* time. In this context we will also rephrase our main result in terms of the *initial-value* problem for (1.3).

As shown below, the main finding reported in this paper is a simple consequence of the change of independent variable that relates (1.1) to (1.3). This 'trick', which was originally noted in the context of a specific many-body problem [4], has recently been shown to be applicable in a much more general context, namely to entire classes of nonlinear evolution equations (both ODEs and PDEs), to yield modified versions of them that possess lots of periodic solutions [5–10]. The main reason to apply it here to (1.1) is because of the historical relevance of this ODE [1, 2], and moreover because it is instructive to verify in the context of this single, solvable ODE—or rather of a special case of it, see below—the type of singularity analysis that we apply elsewhere [10], the conclusions of which, concerning the periodicity of modified equations, can in the present case be confirmed by taking advantage of the availability of the *general* solution in explicit closed form (see (1.2) and below).

In section 2 the 'trick' is introduced, and our main result is proven and discussed in the context of the initial-value problem for the evolution ODE (1.3). In the following section 3 two simpler nonlinear ODEs are considered (corresponding essentially to (1.3) with $k = 1$ respectively $k = 0$), and in section 4 the analysis used elsewhere in a more general context [10] is applied to the first of these simplified equations and it is shown to predict correctly the periodicity properties indeed displayed by its general solution.

2. Main result

We exploit the following change of (independent) variable:

$$f(t) = w(z) \tag{2.1a}$$

$$z = [\exp(\mathrm{i}\omega t) - 1]/(\mathrm{i}\omega) \tag{2.1b}$$

which clearly entails

$$\dot{f}(t) = \exp(\mathrm{i}\omega t)\, w'(z) \tag{2.2a}$$

$$\ddot{f} - \mathrm{i}\omega\, \dot{f}(t) = \exp(2\mathrm{i}\omega t)\, w''(z) \tag{2.2b}$$

hence clearly transforms (1.1) into (1.3). Note that this change of independent variable, (2.1), also entails the following simple relations between the 'initial data' for (1.1) and (1.3):

$$f(0) = w(0) \qquad \dot{f}(0) = w'(0). \tag{2.3}$$

It is therefore clear, see (1.2), that the *general* solution of (1.3) reads

$$f(t) = \mathrm{sn}(\lambda \,\log\{[a\, \exp(\mathrm{i}\omega t) - b]/(\mathrm{i}\omega)\}, k) \tag{2.4a}$$

with $a = -\mathrm{i}A/\omega$, $b = B - \mathrm{i}A/\omega$ two *a priori* arbitrary constants, related to the initial data as follows:

$$f(0) = \mathrm{sn}(\lambda\, \log[(a-b)/(\mathrm{i}\omega)], k) \tag{2.5a}$$

$$\dot{f}(0) = \mathrm{i}\omega\,\lambda\, a\,(a-b)^{-1}\,\{1 - [f(0)]^2\}^{1/2}\{1 - k^2[f(0)]^2\}^{1/2}. \tag{2.5b}$$

It is now clear from (2.4*a*) that the solution $f(t)$ of (1.3), considered as a function of the *real* variable t, has the following behaviour. If

$$|a| < |b| \tag{2.6a}$$

then $f(t)$ is periodic in t with period T, see (1.4) and (1.5); and it is clear from (2.5) that the solutions $f(t)$ of (1.3) featuring this periodicity property emerge from a set of initial data $f(0)$, $\dot{f}(0)$ having *nonvanishing* measure in the phase space of such initial data (because (2.6*a*) is an *inequality*). If instead

$$|a| > |b| \tag{2.6b}$$

then it is convenient to rewrite the solution (2.4*a*) as follows:

$$f(t) = \mathrm{sn}(\mathrm{i}\lambda\,\omega t + \lambda\,\log\{[a - b\, \exp(-\mathrm{i}\omega t)]/(\mathrm{i}\omega)\}, k) \tag{2.4b}$$

and this shows that in this case (2.6*b*) the solution $f(t)$ is not periodic in t, unless (1.6*a*) or (1.6*b*) hold, in which case $f(t)$ is again periodic in t, but with period $\tilde{T} = q\,T$ (see table 5 in section 13.16 of [3]). Clearly the two sets of initial data that correspond (via (2.5)) to the *inequalities* (2.6*a*) respectively (2.6*b*) are *complementary*, and they exhaust the entire phase space of initial data $f(0)$, $\dot{f}(0)$ except for a *lower-dimensional* set of data characterized (via (2.5)) by the *equality*

$$|a| = |b|. \tag{2.6c}$$

To this lower-dimensional set of initial data there corresponds a solution (2.4) that becomes singular at the *real* time t_s defined mod(T) by the formula

$$t_s = -(\mathrm{i}/\omega)\, \log(b/a). \tag{2.7}$$

Note, however, that even solutions which emerge from either one of the two complementary sets of initial data that correspond via (2.5) to the two inequalities (2.6*a*), (2.6*b*) (namely, that do *not* satisfy via (2.5) the equality (2.6*c*)) might become singular (in the guise of hitting a pole at $t = t_p \bmod (T)$) if the equation

$$a\ \exp(\mathrm{i}\omega t_p) = b + \mathrm{i}\omega\ \exp\{[2mK(k) + (2n+1)\mathrm{i}K'(k)]/\lambda\} \tag{2.8}$$

admits a *real* solution t_p for some values of the two (arbitrary) *integers* m, n (here of course, as in (1.6), $K(k)$ and $K'(k)$ are the complete elliptic integrals, see section 13.8 of [3], and we rely again on the information in table 5 of section 13.16 of [3]).

3. Special cases

For $k = 1$ the ODE (1.3) takes the simpler (analytic!) form

$$\ddot{g} - \mathrm{i}\omega\,\dot{g} = \dot{g}^2(2g+1)/(g^2 - \lambda^2) \tag{3.1}$$

where for notational convenience we set $g = \lambda\, f$. The *general* solution of this ODE reads

$$g(t) = \lambda(u-1)/(u+1) \tag{3.2a}$$

$$u = \{[a\ \exp(\mathrm{i}\omega t) - b]/(\mathrm{i}\omega)\}^{2\lambda} \tag{3.2b}$$

and it entails the following relations among the two (*a priori* arbitrary) constants a, b and the initial data $g(0)$, $\dot{g}(0)$:

$$a = \dot{g}(0)\,[\lambda^2 - g^2(0)]^{-1}[\lambda + g(0)]^{1/(2\lambda)}[\lambda - g(0)]^{-1/(2\lambda)} \tag{3.3a}$$

$$b = \{-\mathrm{i}\omega + \dot{g}(0)\,[\lambda^2 - g^2(0)]^{-1}\}[\lambda + g(0)]^{1/(2\lambda)}\,[\lambda - g(0)]^{-1/(2\lambda)}. \tag{3.3b}$$

It is again clear that the solution (3.2) is periodic in t with period T, see (1.4) and (1.5), if $|a| < |b|$, and that it is *not* periodic in t if instead $|a| > |b|$ unless $2\lambda = p/q$ with p and q coprime *integers* (and, say, q positive), in which case $g(t)$, see (3.2), is periodic in t with period $\tilde{T} = q\,T$. Clearly each of these two regimes corresponds, via (3.3), to a set of initial data having nonvanishing measure in the phase space of such initial data. These two sets of initial data are separated by a lower-dimensional set of such data characterized, via (3.3), by the equality $|a| = |b|$.

Likewise, for $k = 0$ equation (1.3) takes the simpler (but still nonanalytic) form

$$\ddot{f} - \mathrm{i}\omega\,\dot{f} = -\dot{f}^2(1-f^2)^{-1}[f + \lambda^{-1}(1-f^2)^{1/2}]. \tag{3.4}$$

The general solution of this ODE reads

$$f(t) = (v - v^{-1})/(2\mathrm{i}) \tag{3.5a}$$

$$v = \{[a\exp(\mathrm{i}\omega t) - b]/(\mathrm{i}\omega)\}^{\mathrm{i}\lambda}. \tag{3.5b}$$

The two constants a, b are now related to the initial data as follows:

$$f(0) = \{[(a-b)/(\mathrm{i}\omega)]^{\mathrm{i}\lambda} - [(a-b)/(\mathrm{i}\omega)]^{-\mathrm{i}\lambda}\}/(2\mathrm{i}) \tag{3.6a}$$

$$\dot{f}(0) = (\lambda a/2)\{[(a-b)/(\mathrm{i}\omega)]^{\mathrm{i}\lambda-1} + [(a-b)/(\mathrm{i}\omega)]^{-\mathrm{i}\lambda-1}\}. \tag{3.6b}$$

Clearly the phenomenology described above, as regards the behaviour of the solutions as functions of the *real* independent variable t, applies again, up to modifications the detailed analysis of which is left as an easy exercise for the diligent reader.

4. Determination of periodic behaviour via branch point analysis

In this section we show how information on the *periodic* behaviour, as function of the real 'time' variable t, of solutions $g \equiv g(t)$ of the (analytic) nonlinear ODE (3.1)

$$\ddot{g} - \mathrm{i}\omega\dot{g} = \dot{g}^2(2g+1)/(g^2 - \lambda^2) \tag{4.1}$$

can be obtained from an analysis of the *analytic structure* of the solutions of the ODE related to it via the transformation (2.1), namely from the ODE

$$w'' = w'^2(2w + 1)/(w^2 - \lambda^2). \tag{4.2}$$

Our motivation for performing this exercise is not to rediscover findings which have been already conclusively established in the preceding section 3, but rather to verify that the route used in this section leads to the correct results—a conclusion we consider significant to sustain our confidence in analogous findings arrived at by analogous treatments in other contexts where, in contrast to the present case, exact explicit solutions are not available to confirm the validity of the findings obtained in this manner [10].

Let us rewrite in the present notation the change of (independent) variable, analogous to (2.1), which relates (4.1) to (4.2),

$$g(t) = w(z) \tag{4.3a}$$

$$z = [\exp(\mathrm{i}\omega t) - 1]/(\mathrm{i}\omega) \tag{4.3b}$$

as well as the relations analogous to (2.3) among the initial data for (4.1) and (4.2),

$$g(0) = w(0) \qquad \dot{g}(0) = w'(0). \tag{4.4}$$

It is clear from these formulae that to solve the initial-value problem for (4.1) one can solve, essentially with the *same* initial data, see (4.4), the initial-value problem for (4.2), obtain thereby the solution $w(z)$, and then use (4.3) to get the solution $g(t)$. One then notes that, as the *real* 'time' variable t varies from 0 to T (see (1.4)), the *complex* variable z travels full circle over the circular contour $\tilde{C}$ the diameter of which lies on the upper imaginary axis of the complex z-plane, with its lower end at the origin, $z = 0$, and its upper end at the point $z = 2\mathrm{i}/\omega$. This clearly entails that, if the solution $w(z)$ so obtained is *holomorphic* (or in fact just *meromorphic*) inside the disc C encircled by $\tilde{C}$, the corresponding solution $g(t)$ is *periodic* in t with period T. Likewise, if the solution $w(z)$ is *not* meromorphic in C but the only singularity (other than poles) it features inside C is a *rational* branch point, namely a singularity at $z = z_b$ (with $|z_b - \mathrm{i}\omega| < 1/\omega$) where the solution $w(z)$ behaves proportionally to $(z - z_b)^{p/q}$ with p, q two coprime *integers* (and, say, q positive, $q = 1, 2, \ldots$), then clearly the corresponding solution $g(t)$, considered as a function of the *real* variable t, is again *periodic*, but now with period $\tilde{T} = q\,T$ (because associated to this branch point is a cut that gives access to a q-sheeted Riemanian surface, hence by encircling it q times one gets again back to the point of departure).

Hence the importance of ascertaining the nature of the singularities of $w(z)$. To this end we note first of all that, because of the structure of the right-hand side of the nonlinear ODE (4.2), at any singular point $z = z_b$ the solution must take the value

$$w(z_b) = s\,\lambda \qquad s^2 = 1 \tag{4.5}$$

since only at such values of w the nonlinear ODE (4.2) can develop a singularity. And secondly, to analyse the type of singularity featured by a solution $w(z)$ of the ODE (4.2) we turn this ODE into a recursion by writing

$${}^{(j+1)}w'' = {}^{(j)}w'^2(2\,{}^{(j)}w + 1)/({}^{(j)}w^2 - \lambda^2) \qquad j = 1, 2, \ldots \tag{4.6}$$

setting moreover (consistently with (4.5))

$${}^{(0)}w(z) = s\lambda + c(z - z_b)^\gamma \tag{4.7a}$$

which clearly also entails

$${}^{(0)}w'(z) = c\gamma(z - z_b)^{\gamma-1} \tag{4.7b}$$

$${}^{(0)}w''(z) = c\gamma(\gamma - 1)(z - z_b)^{\gamma-2} \tag{4.7c}$$

with the constants c and (especially) γ to be determined *a posteriori* so that the leading behaviour of the subsequent terms yielded by the recursion (4.6) be consistent with (4.7). Note that the requirement that (4.7a) be consistent with (4.5) entails the condition

$$\mathrm{Re}\,(\gamma) > 0. \tag{4.8}$$

Insertion of (4.7) in (4.6a), (4.6b) yields

$$\begin{aligned} {}^{(1)}w''(z) &= c\,\gamma^2\,[1 + (2s\lambda)^{-1}](z - z_b)^{\gamma-2} \\ &\quad\times[1 + 2c(1 + 2s\lambda)^{-1}\,(z - z_b)^{\gamma}]\,[1 + c(2s\lambda)^{-1}(z - z_b)^{\gamma}]^{-1} \end{aligned} \tag{4.9a}$$

$${}^{(1)}w''(z) = c\,\gamma^2\,[1 + (2s\lambda)^{-1}](z - z_b)^{\gamma-2}\left[1 + \sum_{l=1}^{\infty}{}^{(1)}\alpha_l\,(z - z_b)^{l\gamma}\right]. \tag{4.9b}$$

The definition of the coefficients ${}^{(1)}\alpha_l$ in (4.9b) is sufficiently obvious from (4.9a) not to require explicit display.

We now require that the dominant term in the right-hand side of (4.9b) reproduce the right-hand side of (4.7c). This entails no condition on the constant c, while it requires validity of the relation

$$\gamma[1 + (2s\lambda)^{-1}] = \gamma - 1 \tag{4.10a}$$

which fixes the value of the exponent γ,

$$\gamma = -2s\lambda. \tag{4.10b}$$

For any given λ (note that the ODE (4.1) only contains λ^2, hence λ is defined up to a sign) this formula, (4.10b), together with (4.8), also fixes the value of the sign s, via the requirement

$$\mathrm{Re}\,(s\lambda) < 0. \tag{4.11}$$

We now integrate twice (4.9b) (with (4.10)), fixing the two integration constants via the requirement that the dominant terms in the right-hand side of the resulting expressions, first of ${}^{(1)}w'(z)$ and then of ${}^{(1)}w(z)$, reproduce the right-hand sides firstly of (4.7b) and then of (4.7a). We thus get

$${}^{(1)}w(z) = s\lambda + c\,(z - z_b)^{\gamma}\left[1 + \sum_{l=1}^{\infty}{}^{(1)}\tilde{\alpha}_l\,(z - z_b)^{l\gamma}\right] \tag{4.12a}$$

with

$${}^{(1)}\tilde{\alpha}_l = {}^{(1)}\alpha_l\,(\gamma - 1)\,/\,\{\,(l + 1)\,[(l + 1)\,\gamma - 1]\}$$

namely

$${}^{(1)}w(z) = s\lambda + c\,(z - z_b)^{\gamma} + \sum_{l=1}^{\infty}{}^{(1)}\beta_l\,(z - z_b)^{(l+1)\,\gamma} \tag{4.12b}$$

with ${}^{(1)}\beta_l = c\,{}^{(1)}\tilde{\alpha}_l$, the exponent γ in these equations (4.12) being of course fixed by (4.10b) with (4.11). Note that this expression, (4.12b), of ${}^{(1)}w(z)$ contains two *a priori* arbitrary constants, namely z_b and c, while the coefficients ${}^{(1)}\beta_l$ are clearly determined (indeed, in explicit form) in terms of these two constants (as well of course as of the parameter λ).

Insertion of this expression in (4.6) with $j = 1$ leads, via a completely analogous integration procedure, to a completely analogous formula,

$${}^{(2)}w(z) = s\lambda + c\,(z - z_b)^{\gamma} + \sum_{l=1}^{\infty}{}^{(2)}\beta_l\,(z - z_b)^{(l+1)\,\gamma} \tag{4.12c}$$

where the coefficients ${}^{(2)}\beta_l$ are again uniquely defined in terms of the three constants z_b, c and λ (although no more in explicit closed form, but only via a recursive procedure yielding the successive determination of the coefficients of power expansions).

It is thereby seen that by successive iterations, see (4.6), one arrives at the formula

$$ {}^{(j)}w(z) = s\lambda + c\,(z - z_b)^{-2s\lambda} + \sum_{l=1}^{\infty} {}^{(j)}\beta_l\,(z - z_b)^{-2\,(l+1)\,s\lambda} \tag{4.13} $$

now valid for all values of j, $j = 1, 2, \ldots$ (and note that in this formula we have also replaced the exponent γ with its expression (4.10b)); and by taking the limit $j = \infty$ (whereby (4.6) coincides with (4.2), leading to the identification ${}^{(\infty)}w(z) = w(z)$) we arrive finally at the following expression for the solution $w(z)$ of (4.2) having a singularity at $z = z_b$:

$$ w(z) = s\lambda + c\,(z - z_b)^{-2s\lambda} + \sum_{l=1}^{\infty} \beta_l\,(z - z_b)^{-2\,(l+1)\,s\lambda}. \tag{4.14} $$

As entailed by its derivation, this expression, (4.14), contains the two arbitrary constants z_b and c (hence it qualifies as an expression of the *general* solution of the second-order ODE (4.2)), while the coefficients β_l appearing in its right-hand side are uniquely determined in terms of these two constants (and of the parameter λ featured by the ODE (4.2)) via a constructive procedure (based on the insertion of this *ansatz* (4.14) into (4.2)) that allows in principle to determine as many of these coefficients as one wishes (sequentially for $l = 1, 2, \ldots$), although it does not provide a closed form expression for them. Let us also recall that the sign s must be chosen consistently with the condition (4.11), which is indeed essential to ensure the consistency of (4.14) with (4.5), as well as the presumed validity of (4.14) as a representation of the solution of (4.2) in the neighbourhood of $z = z_b$.

This finding allows to conclude (although it does not quite prove) that at the singular point $z = z_b$ the solution $w(z)$ of (4.2) possesses a branch point characterized by the exponent $2\,\lambda$, with λ uniquely determined by the parameter λ^2 featured by the ODE (4.2) via the additional requirement that $\mathrm{Re}\,(\lambda)$ be *positive*, $\mathrm{Re}\,(\lambda) > 0$ (entailing, in the formulae written above, the assignment $s = -1$, see (4.11)); with the consequential implications detailed above as regards the periodicity properties of the corresponding solution of (4.1).

These conclusions are confirmed by the explicit *general* solution of (4.2) which (consistently with the results of section 3) reads

$$ w(z) = -\lambda\,(1 - u)\,/\,(1 + u) \tag{4.15a} $$

$$ u = (A\,z - B)^{2\lambda} \tag{4.15b} $$

with A, B two arbitrary constants. Indeed this formula, (4.15), is clearly consistent with the expansion (4.14), and of course it implies $z_b = B/A$ as well as $c = 2\,\lambda\,A^{2\lambda}$ (where we are again assuming that $\mathrm{Re}\,(\lambda)$ is *positive*, $\mathrm{Re}\,(\lambda) > 0$).

Let us end by noting that the analysis given above gets into some ambiguity if λ^2, see (4.1) and (4.2), is *real* and *negative*, entailing that λ is *imaginary*, hence that $\mathrm{Re}\,(\lambda)$ cannot be chosen *positive* because it vanishes, $\mathrm{Re}\,(\lambda) = 0$. This is indeed—up to trivial notational changes—the case that is singled out in [3] (see the very first equation of chapter XIV) as example of an equation whose solution possesses an essential singularity which is 'movable', i.e. located at the *a priori* arbitrary position $z = z_b = B/A$, see (4.15) (although I would rather say it possesses there a branch point with purely imaginary exponent).

References

[1] Painlevé P 1900 Mémoire sur les équations differentielles dont l' intégral est uniforme *Bull. Soc. Math. France* **28** 201–61

[2] Ince E L 1956 *Ordinary Differential Equations* (New York: Dover) (see in particular section 14.39)

[3] Erdélyi A (ed) 1953 *Higher Transcendental Functions* vol 2 (New York: McGraw-Hill) (see in particular sections 13.16 and 13.8)

[4] Calogero F 1997 A class of integrable Hamiltonian systems whose solutions are (perhaps) all completely periodic *J. Math. Phys.* **38** 5711–19

[5] Calogero F and Françoise J-P 2001 Periodic solutions of a many-rotator problem in the plane *Inverse Problems* **17** 1–8

[6] Calogero F 2001 *Classical Many-Body Problems Amenable to Exact Treatments (Lecture Notes in Physics Monograph m 66)* (Berlin: Springer)

[7] Calogero F 2001 Differential equations featuring many periodic solutions *Proc. Istanbul Course* at press

[8] Calogero F and Françoise J-P 2002 Periodic motions galore: how to modify nonlinear evolution equations so that they feature a lot of periodic solutions *J. Nonlinear Math. Phys.* at press

[9] Calogero F 2001 Periodic solutions of a system of complex ODEs *Phys. Lett.* A at press

[10] Calogero F, Françoise J-P and Sommacal M 2001 Periodic solutions of a many-rotator problem in the plane: II. Analysis of various motions *J. Nonlinear Math. Phys.* submitted